The Quantum Theory And Statistical Inference

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Abstract

From a statistical point of view the theory of quanta introduced a revolutionary view of the statistics of radiation energy, still unrecognized as such. It took advantage of a relationship between two descriptors related to energy, namely its mean and its variance. That relationship came to be explicitly recognized in theoretical statistics only as late as the second half of the 20th century. It helps us in understanding that quantization is first and foremost an approach of statistics, which is needed in physics. The meaning of that approach is discussed, and then it is shown that it has deep implications in theoretical physics, still to be considered.

Key words: Statistical Inference, Quantum Theory, Physics.

INTRODUCTION

The fact that statistical physics can be taken as a particular type of statistical inference does not seem to play too much of a theoretical role in physics today. It is perhaps the strong suggestion of subjectivity associated with the concept of inference that determines physicists to rely mostly upon kinetic basis of statistical laws. However, recognized or not, the statistical inference has played a very important part in such fundamental problems like building physical concepts. One illustrative example is the concept of quantum. Originally related to the frequency property of light, it was in time explained as associated with its particle properties, then with the temperature of light. All these properties can be traced logically back to the particular type of statistics which is fundamental for the ensembles characterizing the black body radiation. More than this, that very type of statistics is essential in the contemporary sophisticated descriptions of the squeezed states and their related concepts. This essay presents details of the statistics involved in the history of quantum, and the essential points of this history.

First Example of Quadratic Variance Distributions

In building quantum statistics for the blackbody radiation, Planck had at his disposal the two formulas for the spectral density of radiation, corresponding to the limit cases of high and low temperatures. Both these cases satisfy the Wien's displacement law, which is the only theoretical criterion for the choice of radiation laws. According to Born (Born, 1955), Planck's first reasoning can be linked to the properties of the Gaussian probability distribution. It is worth going into detail along this path, inasmuch as it gives us the overall idea of this important gnoseological process. Born's starting point is, like Planck's, the energy density as a function of temperature, with its extreme cases

$$u(\beta) = \begin{cases} \beta^{-1} & \text{for} \quad T \to \infty \\ u_0 e^{-\varepsilon_0 \beta} & \text{for} \quad T \to 0 \end{cases} \qquad (1)$$

where β ≡ (kT)-1, with k the Boltzmann constant, T the absolute temperature and ε0 an energy which has to be proportional with the frequency of light in order to satisfy the requirement of Wien's displacement law. Then Born proceeds to the first of Planck's steps, which was to study the entropy of such a system, which he assimilated with a system of oscillators. The rationale should have been like this: the entropy is classically related to heat exchanged, at equilibrium, between two thermodynamic systems, or between a system and the Universe. Here we have the heat in the form of thermal radiation! Nothing more natural then, than considering the heat as determined by the energy of this radiation. According to Born, Planck's intention has been facilitated by the important discovery that the coefficient u'(β) (prime denoting the derivative with respect to variable) has a simple statistical meaning. This can be obtained starting from Thermodynamics. Indeed, as the radiation just represents the heat exchanged in equilibrium, in the formula defining the entropy:

$$dS = \frac{\delta Q}{T} \tag{2}$$

we only have to identify the amount of heat with (du) the differential of the density of radiation energy. In so doing, equation (2) can be rewritten as

$$dS = k\beta \cdot du \quad \therefore \quad S'(u) = k\beta \tag{3}$$

At this point one has to recall the Einstein's procedure whereby one identifies the thermodynamic entropy with the statistical entropy as related to the probability by the Boltzmann relation:

$$S = -k \cdot \ln(P) \quad \therefore \quad P = e^{-\frac{S}{k}} \tag{4}$$

Here two arguments have to be used: the energy as carried by radiation is actually characterized by fluctuations and, as these fluctuations take place at equilibrium, the entropy has to be maximum according to classical precepts. Then the entropy, as a function of energy density can be expanded as

$$S(u) = S(u_0) + S'(u_0) \cdot \Delta u + \frac{1}{2} S''(u_0) \cdot (\Delta u)^2 + \ldots \tag{5}$$
$$\approx S_0 + \frac{1}{2} S''(u_0) \cdot (\Delta u)^2$$

and the Boltzmann formula reveals (although approximately) a Gaussian describing the fluctuations represented by the thermal radiation. We write it in the normalized form as:

$$P_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{6}$$

where X ≡ Δu and σ2 is the variance of this process which, with equation (3), can be written in the form

$$\sigma^2 \equiv \left(S''(u)\right)^{-1} = u'(\beta) \tag{7}$$

provided u is a continuous function of temperature. In other words, the first derivative of the energy density with respect to the inverse temperature is actually the variance of a normal distribution characterizing the fluctuations of energy of the field representing the thermal radiation. This is a general property of the exponential distributions, the class of distributions mainly used in statistical physics (**Lavenda, 1992**)

This was the statistical meaning apparently revealed by Planck, and the reason he insisted upon a close consideration of the equation (1) which, in view of this, can be rewritten as (**Born, 1955**)

$$u'(\beta) = \begin{cases} -u^2 & \text{for} \quad T \to \infty \\ -\varepsilon_0 u & \text{for} \quad T \to 0 \end{cases} \tag{8}$$

Actually Planck worked directly with the entropy (Planck, 1900). As mentioned above, considering the energy density is the mark of an Einstein-Born style approach of the problem, but it closely parallels Planck's own way having, besides this, a hint towards the newest theoretical discoveries in the problem of radiation.

Now, keeping in mind equation (7), which shows that we are actually looking at the variance of a Gaussian process, the equation (8) can be interpreted as representing two copies of this process, for the cases of high and low temperatures. For the instance where these two processes are statistically independent, the variance of the compound process is the sum of the two component variances, so one can infer that, in general, the law of radiation could be represented by the following differential equation:

$$u'(\beta) = -u^2 - \varepsilon_0 u \tag{9}$$

This is what Planck really did. This equation has an immediate particular solution

$$u(\beta) = \frac{\varepsilon_0}{e^{\beta\varepsilon_0} - 1} \tag{10}$$

and from this moment on everything is recorded history. Mention should be made that the general solution depends on one arbitrary constant leading to the Bose statistics, which was revealed later on.

The equation (9) is the first example of an equation characterizing what in modern terms is known as a quadratic variance (**Morris, 1982**): the variance of the process is a quadratic polynomial in its mean. This type of distribution has interesting properties, many of them still to be discovered and studied. But these facts were not known at the moment when the blackbody radiation was studied by Planck, so he used the heuristic approach illustrated above. As for the physical interpretation of the result (10), an apparently happy coincidence illuminated at once its obscurity. And even though there have been many other proposals regarding the formula of energy distribution of the blackbody radiation ever since, that illumination made Planck interpretation outstanding, in fact a truth.

Fact is that equation (10) has a physical interpretation unveiling the statistical ensemble that has it for a mean. The physical interpretation goes on describing an ensemble of harmonic oscillators, each one having the energy an integer multiple of ε0, of which we know nothing but that it is an energy – in order to make the equation (9) physically meaningful – which is proportional with the frequency – in order to make the equation (10) physically meaningful. As mentioned before the existence of this energy is enforced by the fact that the density of thermal spectrum has to satisfy Wien's displacement law. The partition function of this ensemble can be written in terms of Boltzmann factors of its constitutive oscillators as

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n\beta\varepsilon_0} = \frac{1}{1 - \exp(-\beta\varepsilon_0)} \tag{11}$$

so u(β) from equation (10) appears indeed as the mean of this ensemble, according to the usual formalism of statistical mechanics (**Lavenda, 1992**):

$$u(\beta) \equiv \frac{\partial}{\partial\beta} \ln Z(\beta) \tag{12}$$

The presence of quantum is here suggested by the random variable characterizing this distribution, which is a natural integer. The photon comes into play later on. However, it seems that the modern theories rely mostly upon what has been left behind along the way towards quantum mechanics, which started from this beginning. Let us bring some illustrations.

Introducing Correlations: an Interpretation of Quantum

Let us limit considerations first to the Gaussian realm of statistical distributions: according to previous plot of Born everything is based on the normal distribution. Moreover, it seems that the Planck's pursuit was exclusively for statistically independent processes. One can easily be induced to think that the two processes characterizing the light might not be independent. Assume that we have indeed two Gaussian processes representing the limit cases of radiation, but they are in a general relationship, i.e. not statistically independent. One may further assume that the general radiation is actually a linear combination between the two processes, but we limit here the line of reasoning to just the sum of the two processes, like Planck. The general bivariate normal distribution is given by

$$P_{XY}(x, y) = \frac{\sqrt{ac - b^2}}{2\pi} \exp\left\{-\frac{1}{2}(ax^2 + 2bxy + cy^2)\right\} \tag{13}$$

In terms of the variances σx, σy of the two component processes and their correlation coefficient r, the coefficients a, b, c can be written as

$$a^{-1} = \sigma_x^2(1-r^2), \quad b^{-1} = -\sigma_x\sigma_y(1-r^2), \quad c^{-1} = \sigma_y^2(1-r^2) \tag{14}$$

Now we can write the probability density of the compound process (X + Y), which is

$$P_{X+Y}(\xi) = \sqrt{\frac{ac-b^2}{2\pi(a+c-2b)}} \exp\left\{-\frac{1}{2}\frac{ac-b^2}{a+c-2b}\xi^2\right\} \tag{15}$$

i.e. a Gaussian with the variance (a + c – 2b)/(ac – b2) or, in terms of variances and correlation coefficient of the two component processes

$$\sigma_\xi^2 = \sigma_x^2 + \sigma_y^2 + 2r\sigma_x\sigma_y \tag{16}$$

Still maintaining the philosophy above, instead of equation (9) we should have

$$u'(\beta) = -u^2 - \varepsilon_0 u - 2r\sqrt{\varepsilon_0}\, u\sqrt{u} \tag{17}$$

This equation can be integrated to give

$$\beta\varepsilon_0 = \ln(1+2rw+w^2) - 2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\frac{1-rw}{w\sqrt{1-r^2}}, \qquad w^2 \equiv \frac{\varepsilon_0}{u} \tag{18}$$

Here something is immediately obvious, which shows that, in the past, our focus might have been misled by the mirage of quick statistical interpretation already in hand. Namely the energy ε0, which was introduced from dimensional considerations, and which has subsequently been explained as a quantum of energy to be carried by an invented particle whose existence is nowadays challenged, had to be a priori explained. Here is a scenario: in the good old fashion of statistical mechanics, we can correlate this energy with an exponential factor, which can play the role of a partition function over a certain ensemble, and which can be easily extracted from equation (18) as

$$e^{-\beta\varepsilon_0} = \frac{1}{1+2rw+w^2}\cdot\exp\left[2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\frac{w\sqrt{1-r^2}}{1-rw}\right] \tag{19}$$

The left hand side of this equation represents a thermal ensemble for the energy ε0, having the mean β. The odd thing here is that the right hand side also depends on ε0. However, this dependence occurs through the intermediary of the ratio w, which allows us to say that a statistical interpretation actually depends upon a sort of "ε0-content" of the density of energy of the thermal radiation. This conclusion sounds quite normal: an experimentalist knows exactly how to characterize the radiation depending on its density. Should this density be of the order of ε0 then w ≈ 1, and the right hand side of equation (19) does not depend but on the correlation coefficient between the two processes:

$$e^{-\beta\varepsilon_0} = \frac{1}{2(1+r)}\cdot\exp\left[2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\sqrt{\frac{1+r}{1-r}}\right] \tag{20}$$

Three decades ago, Ioannidou (Ioannidou, 1982) tried to explain the quantum through the correlation of ensembles associated with oscillators, based on the uncertainty relation. The attempt has been forgotten, probably due to the connection it suggested. Well, that connection seems sound, for here it is again, in equation (20), which explicitly sets down a relationship between the quantum and the correlation coefficient of the two processes representing the radiation. Further on, if the correlation of the two processes is faint, which is the Planck's case, then

$$\varepsilon_0 = (\ln 2)\cdot kT \tag{21}$$

independent of any other consideration. Thus, in this limit, the "quantum", and therefore the frequency, is directly proportional to the temperature. This fact has been discussed at length by Louis de Broglie (De Broglie, 1964), who skillfully identified the action with the entropy.

In the general case, when the ε0-content units and the correlation of the two processes are both arbitrary, it helps noticing that the right hand side of (19) is actually the generating function of a particular class of Pollaczek polynomials (Chihara, 1978). Specifically, we can write (19) in the form

$$e^{-\beta\ _0} = \sum_{n=0}^{\infty} P_n^1(r)\cdot w^n \tag{22}$$

The orthogonality relation of the polynomials involved here is given by

$$\int_{-1}^{1} P_n^a(r)P_m^a(r)\rho(r)dr = \frac{\pi}{2}\frac{n+1}{n+a+1}\delta_{mn} \tag{23}$$

with the weight function ρ given by

$$\rho(r) = \sin\theta \cdot e^{(2\theta-\pi)\tan\theta}\left|\Gamma(1+i\tan\theta)\right|^2, \quad r \equiv \cos\theta; \theta \in [0,\pi] \tag{24}$$

where Γ is the Euler function of the first kind, generalization of the factorial. Should we agree to interpret ε0 as the energy of a photon, as has historically been the case, then the formula (22) would be the source of constructions of some modern quantum states related to the coherence properties of radiation.

The bottom line is that we ought to supply for ε0 an a priori explanation; otherwise we have to face problems of a posteriori explanations, as indeed was historically the case. One of these explanations aims the very existence of the photon, or of any other 'ons' for that matter. The ongoing discussion on the legitimacy of the quantum precepts (Marshall, Santos, 1997) just proves this conclusion.

Formal Description of the Quadratic Variance Ensembles

By accepting the Boltzmann's and Gibbs' view regarding the relation between probability and entropy, the physics placed itself into the hands of the so-called class of exponential families of distributions (**Lavenda, 1992**). In the case of one statistical variable X such a family, having the elementary probability given by

$$F_\xi(dx) = \left[Z(\xi)\right]^{-1} e^{\xi x}\nu(dx) \tag{25}$$

is known as natural exponential family. Here ξ is the parameter scanning the family, while ν(dx) is a Stieltjes measure of the domain of statistical variable X. The parameter ξ, usually related to the measurement of the variable X, is connected to the mean of the ensembles characterized by (25) through equation

$$m(\xi) = \frac{d}{d\xi}(\ln Z(\xi)) \tag{26}$$

a relation used explicitly before in order to interpret u as a mean over an ensemble of oscillators.

Now, the previous strides of reasoning may be flawed by the fact that the differential equation (9) looks much like an interpolation equation, having no substance of principle involved in its derivation. Its formal deduction as based on Gaussian distribution in the previous section might thus be tarnished as being too particular, if not very approximate. Fact is that one can get the equation for the fluctuations starting from physical considerations upon the field sustaining the fluctuations, and the result is a quadratic polynomial for the variance. This fact proves to be essential from a certain point of view. Indeed, if we limit the considerations to natural exponentials, we then have necessarily

$$m'(\xi) = V(\xi) \tag{27}$$

where V(ξ) is the variance function of the family of exponentials. If this function depends on the parameter in such a way that it can be arranged as a quadratic polynomial in the mean of the family of distributions, then we have a particular case of exponentials, modernely termed as quadratic variance distribution functions (**Morris, 1982**). These distributions cover just about everything we use today in the realm of Physics and Engineering: Gaussian, Binomial, Negative Binomial, Poisson, Gamma, Generalized Hyperbolic Secant.

Then, one can notice the advantage of the Born's approach to Planck's problem even if we do not use the Gaussian approach: in principle, the statistics of radiation can be characterized by such a quadratic relationship between the mean and the variance of an ensemble representing the radiation. The corresponding distribution is not necessarily Gaussian. In this case the whole problem of radiation can be treated in its utmost generality, for the two limiting processes are quite different in nature, albeit both quadratic variance processes. Namely, the equation (9) which is the characteristic of Born's line of reasoning can, by a simple metamorphosis, be put in the Morris' canonical form for a quadratic variance distribution function (**Morris, 1982**, Table 1)

$$m'(\theta) = \frac{m^2}{r} + m \tag{28}$$

Here we denoted

$$\theta = -\beta\varepsilon_0 , \qquad r = \frac{\varepsilon_0}{u_0} \tag{29}$$

where u0 is an arbitrary constant energy density. Here the parameter r is not a correlation coefficient anymore. Then, according to Morris' scheme, Planck's result is a Negative Binomial distribution (NB(r, p)) with the density of probability given by

$$P_X(x; r, p) = \frac{\Gamma(x+r)}{\Gamma(r)\cdot x!} p^x q^r \tag{30}$$

where the variable X is discrete, x = 1, 2, … , p ≡ eθ is a probability (the "Boltzmann factor") and q = 1 – p. The mean of this distribution is given by

$$m = \frac{1}{u_0} \frac{\varepsilon_0}{e^{\beta\varepsilon_0} - 1} \tag{31}$$

which is Planck's formula, written however so as to make explicit the occurrence of unit u0: u = mu0. Let us try to find some limit distributions for NB(r, p).

First of all, we are interested in those probabilities p close to one. As we have

$$V(m) = \frac{m^2}{r} + m \tag{32}$$

and by definition

$$p \equiv \frac{m}{m+r} \tag{33}$$

one can directly write

$$V(m) = \frac{m^2}{rp} \xrightarrow[p \to 1]{} \frac{m^2}{r} \tag{34}$$

According to Morris' classification, this represents a Gamma distribution having the density

$$P_X(x; r, \lambda) = \left(\frac{x}{\lambda}\right)^{r-1} \cdot \frac{e^{-\frac{x}{\lambda}}}{\lambda\Gamma(r)}, \qquad \lambda \equiv -\frac{1}{\theta} = \frac{1}{\beta\varepsilon_0} \tag{35}$$

for x real and positive. For r = 1, i.e. ε0 = u0, this distribution is the classical exponential. In general, for finite r, this limit distribution of radiation refers to the case where the mean energy of the ensemble characterizing the heat radiation is high (m → ∞). The temperature of this state cannot be but high too, in order to make p ~ 1. At arbitrary but finite densities of radiation energy this limit distribution is also characteristic for r → 0, which comes down to ε0 << u0, no matter of the relationship between ε0 and kT, i.e. no matter of the value of the mean m. In other words, the process can be a Gamma process in classical as well as in quantum case, depending on the unit we choose for the measurement of the radiation spectrum.

Another limiting case of the general Negative Binomial distribution characterizing the Planck process is that where the probability p is very small. According to the definition of p this happens for low temperatures, so that θ → − ∞, and is realized by an ensemble of very low energy density. In equation (32) the linear term prevails so that, according to Morris' classification the process is a Poisson one as characterized by the probability density

$$P_X(x; \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, \qquad \lambda \equiv e^{-\beta\varepsilon_0} , \qquad m = \lambda \tag{36}$$

for the variable x = 0, 1, 2, … . For arbitrary energy densities, the Poisson limit distribution can also be realized when r → ∞, i.e. when ε0 >> u0, no matter of the relationship between ε0 and kT. Again, just as before, the process can be a Poisson process in classical as well as in quantum case, depending on the unit we choose for the measurement of the radiation spectrum.

CONCLUSIONS

Is there any need for quantum in Physics? Apparently, the only need comes from the convenience offered initially by statistical meaning of the density of blackbody radiation energy as a mean of the energy density of an ensemble of oscillators. This idea has penetrated the theory of blackbody radiation from the very beginning and lasts to this very day. It did not emerge without questions, as it does not dwell today without questions. In an attempt to discern the nature of these questions we traced back the procedure that led to quantum idea, and the conclusions (as well as the question marks) all seem related to the idea of process representing radiation as follows:

If the blackbody radiation is a stochastic Gaussian process, then the fact is that it is composed of two Gaussian processes. If these are independent, then we can have a quantum interpretation and a posteriori explanation of $\varepsilon 0$ as a quantum of energy, as indeed historically was the case. This however, leaves open the identity of the ensemble of oscillators to which the formula for mean is referred. When it comes the time to attach an identity to this ensemble, as in the process of calculation of specific heats of solids for instance, then the problems show up. Case in point: the necessity of introduction of the zero point energy, which has been a challenge for quite some time. Nowadays nobody seems to dispute the idea of quantum, and the reason is clear: there is no fault in the mind process that led to its idea! This process is conceptually clean. It is however the fact that this quantum has a carrier – the photon – that has always been challenged. Here it helps to recall the fact that this carrier does not qualify as a particle in the classical sense of this last concept, for the original identification was done on the determination of ensemble for the particle, not as individual in the mechanical connotation of the word. This fact has always been left aside in critical discussions, in spite of its overwhelming importance for, had it been taken explicitly into consideration, the concept of photon would have never made its appearance in physics. Anyway, carrier or not carrier, the quantum itself had to be explained. The explanation is not easy but, whenever is produced, it always involves an ensemble. The best known instances are the De Broglie's thermodynamics of isolated particle (De Broglie, 1964) and the theory of hidden parameters (Bohm, 1952), this last being a remote descendant of the original solution to the problem of radiation.

Still regarding the process, the kind of rationale that takes us clearly outside the realm of the Gaussian distribution has apparently been inaugurated by Poincaré (Poincaré, 1911), with the result that the quantum has to be seriously taken into consideration. As we have seen this property is a consequence of the fact that the blackbody radiation ensemble is characterized by a quadratic variance distribution function. This fact could not be recognized at the beginning of the last century, for by then there was not even the faintest notion of such distribution functions. It is worth mentioning that the variance property has already been recognized as such by Einstein at the Solvay Congress in 1911 (De Broglie, 1922) or even earlier. Einstein made even the derivative connection for the relationship mean-variance. However, because in those times, as today for that matter, the quadratic variance equation has not been seen as an essential property, no effort has been made towards describing such distributions. Only slowly and by particular chances had these distributions made their entrance, mainly in the field of Statistics, until 1982 when the fundamental work of Carl Morris appeared (Morris, 1982). That work allows us to say that, with respect to blackbody radiation, there is nothing special about Gaussian distributions, they are just as natural in this field as any other distribution. It is a matter of experimental chance that they are just limit distributions for Gamma and Poisson which, in turn, are limiting distributions for Negative Binomials, which were Planck's original distributions. In witness thereof we have today the plethora of intermediate states (Fu, 1996), descendants of coherent and numerical states. There is an intimate connection between these intermediate states and the probability distributions having quadratic variance functions. Indeed, it has been proved by Fu and Sasaki (Fu, Sasaki, 1996) that all the intermediate states can be generated by a process of 'square-rooting' from the probability distributions that give their name. As expected, the Negative Binomial states are 'intermediate' between thermal states and coherent states.

Without diminishing the merits of theoretical statistics, it is however to be recognized that, historically, the quadratic variance property was one of the main assets of statistical physics. Recognizing this might not be too much from a general point of view, but it certainly can give another perspective to established concepts of Physics.

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